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ON GENERALIZED FUBINI POLYNOMIALS

FENG QI

Abstract. In the paper, by virtue of an explicit formula for higher order derivatives of the ratio between two differentiable functions in terms of the Hessenberg determinant, the author presents determinantal expressions and recurrence relations for the generalized Fubini polynomials.

1. Introduction

For $n \ge 0$, let

$$F_n = \sum_{k=0}^{n} k! S(n,k),$$

where $S(n,k)$ denotes the Stirling numbers of the second kind [12]. One calls these numbers F_n the Fubini numbers [5], ordered Bell numbers [7], or geometric numbers [14]. A Fubini number F_n has been interpreted in [7, 14] combinatorially: it counts all the possible set partitions of an n element set such that the order of the blocks matters. In [7], the Fubini numbers F_n were connected with preference arrangements and the recursion for F_n was derived. In [7, 8], the exponential generating function

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}$$

and an asymptotic estimate for F_n were established.

In [14], the Fubini polynomials $F_n(y)$ were defined by

$$F_n(y) = \sum_{k=0}^{n} S(n,k) k! y^k.$$

These polynomials can be generated [14, pp. 735–736] by

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y) \frac{t^n}{n!}.$$

It is clear that $F_n(1) = F_n$. Due to the relation

$$\left(y\frac{\mathrm{d}}{\mathrm{d}\,y}\right)^m \frac{1}{1-y} = \sum_{k=0}^{\infty} k^m y^k = \frac{1}{1-y} F_m\left(\frac{y}{1-y}\right), \quad |y|<1$$

in [2], one also calls $F_n(y)$ the geometric polynomials.

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In [9, Section 2], the Fubini polynomials $F_n(x,y)$ of two variables x, y are defined by means of the generating function

$$\frac{e^{xt}}{1-y(e^t-1)}=\sum_{n=0}^{\infty}F_n(x,y)\frac{t^n}{n!}.$$

It is apparent that $F_n(0,y)=F_n(y)$.

In [10], the bivariate Fubini polynomials $F_n^{(r)}(x,y)$ of order r, generated by

$$\frac{e^{xt}}{[1-y(e^t-1)]^r}=\sum_{n=0}^{\infty}F_n^{(r)}(x,y)\frac{t^n}{n!},\quad r\in\mathbb{N}$$

were studied. It is obvious that $F_n^{(1)}(x,y)=F_n(x,y)$.

The generating functions of F_n, $F_n(y)$, $F_n(x,y)$, and $F_n^{(r)}(x,y)$ remind us to consider the generating function

$$\frac{e^{xt}}{[z-y(e^t-1)]^q}=\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},\quad x,q\in\mathbb{R}$$

and the generalized Fubini polynomials $F_n(x,y,z,q)$ of four variables x, y, z, q. It is clear that, since

$$\frac{e^{xt}}{[z-y(e^t-1)]^q}=\frac{1}{z^q}\frac{e^{xt}}{[1-(y/z)(e^t-1)]^q},$$

we have

$$F_n(x,y,z,r)=\frac{F_n^{(r)}(x,y/z)}{z^r}.$$

In this paper, we will establish some properties of generalized Fubini polynomials $F_n(x,y,z,q)$ and, consequently, derive some known results.

2. A new determiantal expression of the Fubini polynomials

At first, we simply establish a new determinantal expressions of the Fubini polynomials and numbers $F_n(y)$ and F_n respectively.

Theorem 2.1. *The Fubini polynomials $F_n(y)$ for $n\ge0$ can be represented determinantally by*

$$F_n(y)=\begin{vmatrix}
1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & y & -1 & 0 & \cdots & 0 & 0 & 0\\
0 & y & \binom{2}{1}y & -1 & \cdots & 0 & 0 & 0\\
0 & y & \binom{3}{1}y & \binom{3}{2}y & \cdots & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots\\
0 & y & \binom{n-2}{1}y & \binom{n-2}{2}y & \cdots & \binom{n-2}{n-3}y & -1 & 0\\
0 & y & \binom{n-1}{1}y & \binom{n-1}{2}y & \cdots & \binom{n-1}{n-3}y & \binom{n-1}{n-2}y & -1\\
0 & y & \binom{n}{1}y & \binom{n}{2}y & \cdots & \binom{n}{n-3}y & \binom{n}{n-2}y & \binom{n}{n-1}y
\end{vmatrix}. \tag{2.1}$$

Consequently, the Fubini numbers F_n can be represented determinantally by

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \binom{2}{1} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \binom{3}{1} & \binom{3}{2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 1 & \binom{n-2}{1} & \binom{n-2}{2} & \cdots & \binom{n-2}{n-3} & -1 & 0 \\ 0 & 1 & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-3} & \binom{n-1}{n-2} & -1 \\ 0 & 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-3} & \binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}.$$

Proof. Let $u(t)$ and $v(t) \neq 0$ be differentiable functions, let $U_{(n+1)\times 1}(t)$ be an $(n+1)\times 1$ matrix whose elements $u_{k,1}(t) = u^{(k-1)}(t)$ for $1 \leq k \leq n+1$, let $V_{(n+1)\times n}(t)$ be an $(n+1)\times n$ matrix whose elements

$$v_{i,j}(t) = \begin{cases} \binom{i-1}{j-1} v^{(i-j)}(t), & i-j \geq 0; \\ 0, & i-j < 0 \end{cases}$$

for $1 \leq i \leq n+1$ and $1 \leq j \leq n$, and let $|W_{(n+1)\times(n+1)}(t)|$ denote the lower Hessenberg determinant of the $(n+1)\times(n+1)$ lower Hessenberg matrix

$$W_{(n+1)\times(n+1)}(t) = \begin{bmatrix} U_{(n+1)\times 1}(t) & V_{(n+1)\times n}(t) \end{bmatrix}.$$

Then the nth derivative of the ratio $\frac{u(t)}{v(t)}$ can be computed by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[\frac{u(t)}{v(t)}\right] = (-1)^n \frac{\left|W_{(n+1)\times(n+1)}(t)\right|}{v^{n+1}(t)}. \tag{2.2}$$

See [1, p. 40, Exercise 5)], the papers [11, 13] and closely related references. Taking $u(t) = 1$ and $v(t) = 1 - y\big(e^t - 1\big)$ in (2.2) yields

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\frac{1}{1-y(e^t-1)} = \frac{(-1)^n}{[1-y(e^t-1)]^{n+1}}$$

$$\times \begin{vmatrix} 1 & \binom{0}{0}\big[1-y\big(e^t-1\big)\big] & 0 & \cdots & 0 \\ 0 & \binom{1}{0}\big(-ye^t\big) & \binom{1}{1}\big[1-y\big(e^t-1\big)\big] & \cdots & 0 \\ 0 & \binom{2}{0}\big(-ye^t\big) & \binom{2}{1}\big(-ye^t\big) & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & \binom{n-2}{0}\big(-ye^t\big) & \binom{n-2}{1}\big(-ye^t\big) & \cdots & 0 \\ 0 & \binom{n-1}{0}\big(-ye^t\big) & \binom{n-1}{1}\big(-ye^t\big) & \cdots & \binom{n-1}{n-1}\big[1-y\big(e^t-1\big)\big] \\ 0 & \binom{n}{0}\big(-ye^t\big) & \binom{n}{1}\big(-ye^t\big) & \cdots & \binom{n}{n-1}\big(-ye^t\big) \end{vmatrix}$$

$$\to (-1)^n \begin{vmatrix} 1 & \binom{0}{0} & 0 & \cdots & 0 \\ 0 & \binom{1}{0}(-y) & \binom{1}{1} & \cdots & 0 \\ 0 & \binom{2}{0}(-y) & \binom{2}{1}(-y) & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & \binom{n-2}{0}(-y) & \binom{n-2}{1}(-y) & \cdots & 0 \\ 0 & \binom{n-1}{0}(-y) & \binom{n-1}{1}(-y) & \cdots & \binom{n-1}{n-1} \\ 0 & \binom{n}{0}(-y) & \binom{n}{1}(-y) & \cdots & \binom{n}{n-1}(-y) \end{vmatrix}, \quad t \to 0$$

$$= \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}y & -1 & \cdots & 0 \\ 0 & \binom{2}{0}y & \binom{2}{1}y & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & \binom{n-2}{0}y & \binom{n-2}{1}y & \cdots & 0 \\ 0 & \binom{n-1}{0}y & \binom{n-1}{1}y & \cdots & -1 \\ 0 & \binom{n}{0}y & \binom{n}{1}y & \cdots & \binom{n}{n-1}y \end{vmatrix}.$$

The determinantal expression (2.1) is thus proved. The proof of Theorem 2.1 is complete. □

3. Two recurrence relations of the Fubini polynomials

Basing on Theorem 2.1 and the binomial inversion theorem, we now derive two recurrence relations of the Fubini polynomials $F_n(y)$.

Theorem 3.1. *The Fubini polynomials $F_n(y)$ satisfy*

$$F_n(y) = y\sum_{r=0}^{n-1}\binom{n}{r}F_r(y), \quad n \in \mathbb{N} \tag{3.1}$$

and

$$F_n(y) = (-1)^{n+1}(1+y)\left[\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}F_k(y) - \frac{1}{y}\right], \quad n \ge 0. \tag{3.2}$$

Proof. Let $M_0 = 1$ and

$$M_n = \begin{vmatrix} m_{1,1} & m_{1,2} & 0 & \dots & 0 & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & \dots & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \dots & m_{n-2,n-1} & 0 \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \dots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n,1} & m_{n,2} & m_{n,3} & \dots & m_{n,n-1} & m_{n,n} \end{vmatrix}$$

for $n \in \mathbb{N}$. Then the main theorem on page 222 in [3] reads that the sequence M_n for $n \ge 0$ satisfies $M_1 = m_{1,1}$ and

$$M_n = m_{n,n}M_{n-1} + \sum_{r=1}^{n-1}(-1)^{n-r}m_{n,r}\left(\prod_{j=r}^{n-1}m_{j,j+1}\right)M_{r-1}, \quad n \ge 2. \tag{3.3}$$

Applying (3.3) to the determinantal expression (2.1) and simplifying yield the recurrence relation (3.1).

It is well known [4, pp. 143–144] that the binomial inversion theorem reads that the equation

$$s_n = \sum_{k=0}^{n}\binom{n}{k}S_k, \quad n \ge 0$$

holds if and only if the equation

$$S_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}s_k$$

holds for $n \geq 0$, where $\{s_n, n \geq 0\}$ and $\{S_n, n \geq 0\}$ are sequences of complex numbers. The relation (3.1) can be rearranged as

$$\left(1+\frac{1}{y}\right)F_{n+1}(y) = \sum_{r=0}^{n+1}\binom{n+1}{r}F_r(y), \quad n \geq 0.$$

Therefore, we obtain

$$F_n(y) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\left(1+\frac{1}{y}\right)F_k(y) - \frac{(-1)^n}{y}, \quad n \geq 0$$

which can be rewritten as (3.2). The proof of Theorem 3.1 is complete. □

4. A determinantal expression for generalized Fubini polynomials

In this section, We establish a determinantal expression for generalized Fubini polynomials $F_n(x, y, z, q)$ of four variables x, y, z, q.

Theorem 4.1. *Generalized Fubini polynomials $F_n(x, y, z, q)$ of four variables x, y, z, q can be determinantally expressed as*

$$F_n(x,y,z,q) = \frac{(-1)^n}{z^{q(n+1)}}\begin{vmatrix} 1 & \binom{0}{0}z^q & 0 & \cdots & 0 \\ x & \binom{1}{0}m_1(z,y,q) & \binom{1}{1}z^q & \cdots & 0 \\ x^2 & \binom{2}{0}m_2(z,y,q) & \binom{2}{1}m_1(z,y,q) & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ x^{n-2} & \binom{n-2}{0}m_{n-2}(z,y,q) & \binom{n-2}{1}m_{n-3}(z,y,q) & \cdots & 0 \\ x^{n-1} & \binom{n-1}{0}m_{n-1}(z,y,q) & \binom{n-1}{1}m_{n-2}(z,y,q) & \cdots & \binom{n-1}{n-1}z^q \\ x^n & \binom{n}{0}m_n(z,y,q) & \binom{n}{1}m_{n-1}(z,y,q) & \cdots & \binom{n}{n-1}m_1(z,y,q) \end{vmatrix} \tag{4.1}$$

and satisfy the recurrence relation

$$F_n(x,y,z,q) = \frac{2}{z^q}\left[x^n - \sum_{r=0}^{n-1}\binom{n}{r}m_{n-r}(z,y,q)F_r(x,y,z,q)\right], \tag{4.2}$$

where

$$m_k(z,y,q) = \sum_{\ell=0}^{k}(-1)^\ell S(k,\ell)\langle q\rangle_\ell z^{q-\ell}y^\ell, \quad k \geq 0.$$

Proof. In [4, p. 134, Theorem A] and [4, p. 139, Theorem C], the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ for $n \geq k \geq 0$ are defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\ldots,x_{n-k+1}) = \sum_{\substack{1\leq i\leq n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i = n\\ \sum_{i=1}^{n-k+1}\ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t))\,\mathrm{B}_{n,k}\big(h'(t), h''(t), \ldots, h^{(n-k+1)}(t)\big). \tag{4.3}$$

In [4, p. 135], it is given that

$$\mathrm{B}_{n,k}\bigl(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\bigr) = a^k b^n \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{4.4}$$

and

$$\mathrm{B}_{n,k}(1, 1, \dots, 1) = S(n, k), \tag{4.5}$$

where a and b are complex numbers.

By the Faà di Bruno formula (4.3) and the identities (4.4) and (4.5), we obtain

$$\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\bigl[z - y\bigl(e^t - 1\bigr)\bigr]^q &= \sum_{\ell=0}^{k} \frac{\mathrm{d}^\ell\, u^q}{\mathrm{d}\,u^\ell} \mathrm{B}_{k,\ell}\bigl(-ye^t, -ye^t, \dots, -ye^t\bigr) \\
&= \sum_{\ell=0}^{k} \langle q\rangle_\ell u^{q-\ell} \bigl(-ye^t\bigr)^\ell \mathrm{B}_{k,\ell}(1, 1, \dots, 1) \\
&= \sum_{\ell=0}^{k} \langle q\rangle_\ell \bigl[z - y\bigl(e^t - 1\bigr)\bigr]^{q-\ell} \bigl(-ye^t\bigr)^\ell S(k, \ell) \\
&\to \sum_{\ell=0}^{k} (-1)^\ell S(k, \ell) \langle q\rangle_\ell z^{q-\ell} y^\ell
\end{aligned}$$

as $t \to 0$, where $u = u(t) = z - y\bigl(e^t - 1\bigr)$ and

$$\langle q\rangle_n = \prod_{k=0}^{n-1} (q - k) = \begin{cases} q(q-1)\cdots(q-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

is the falling factorial of the number q.

In the formula (2.2), taking $u(t) = e^{xt}$ and $v(t) = \bigl[z - y\bigl(e^t - 1\bigr)\bigr]^q$ yields

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n} \frac{e^{xt}}{[z - y(e^t - 1)]^q} = \frac{(-1)^n}{[z - y(e^t - 1)]^{q(n+1)}}$$

$$\times \begin{vmatrix}
e^{xt} & \binom{0}{0}\bigl(z + y - ye^t\bigr)^q & 0 & \cdots & 0 \\
xe^{xt} & \binom{1}{0}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]' & \binom{1}{1}\bigl(z + y - ye^t\bigr)^q & \cdots & 0 \\
x^2e^{xt} & \binom{2}{0}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]'' & \binom{2}{1}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]' & \cdots & 0 \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
x^{n-2}e^{xt} & \binom{n-2}{0}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]^{(n-2)} & \binom{n-2}{1}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]^{(n-3)} & \cdots & 0 \\
x^{n-1}e^{xt} & \binom{n-1}{0}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]^{(n-1)} & \binom{n-1}{1}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]^{(n-2)} & \cdots & \binom{n-1}{n-1}\bigl(z + y - ye^t\bigr)^q \\
x^ne^{xt} & \binom{n}{0}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]^{(n)} & \binom{n}{1}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]^{(n-1)} & \cdots & \binom{n}{n-1}\bigl[\bigl(z + y - ye^t\bigr)^q\bigr]'
\end{vmatrix}$$

$$\to \frac{(-1)^n}{z^{q(n+1)}} \begin{vmatrix}
1 & \binom{0}{0}z^q & 0 & \cdots & 0 \\
x & \binom{1}{0}m_1(z, y, q) & \binom{1}{1}z^q & \cdots & 0 \\
x^2 & \binom{2}{0}m_2(z, y, q) & \binom{2}{1}m_1(z, y, q) & \cdots & 0 \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
x^{n-2} & \binom{n-2}{0}m_{n-2}(z, y, q) & \binom{n-2}{1}m_{n-3}(z, y, q) & \cdots & 0 \\
x^{n-1} & \binom{n-1}{0}m_{n-1}(z, y, q) & \binom{n-1}{1}m_{n-2}(z, y, q) & \cdots & \binom{n-1}{n-1}z^q \\
x^n & \binom{n}{0}m_n(z, y, q) & \binom{n}{1}m_{n-1}(z, y, q) & \cdots & \binom{n}{n-1}m_1(z, y, q)
\end{vmatrix}$$

as t tends to 0. The determinantal expression (4.1) is thus proved.

Applying (3.3) to the determinant in (4.1) gives

$$\begin{aligned}(-1)^n z^{q(n+1)} F_n(x,y,z,q) &= \binom{n}{n-1} m_1(z,y,q)(-1)^{n-1} z^{qn} F_{n-1}(x,y,z,q)\\ &\quad +(-1)^{n-1+1} x^n z^{qn} (-1)^{1-2} z^{q(1-1)} F_{1-2}(x,y,z,q)\\ &\quad +\sum_{r=2}^{n} (-1)^{n-r+1} \binom{n}{r-2} m_{n-r+2}(z,y,q) z^{q(n-r+1)} (-1)^{r-2} z^{q(r-1)} F_{r-2}(x,y,z,q),\end{aligned}$$

where $F_{-1}(x,y,z,q) = -1$. This can be simplified as (4.2). The proof of Theorem 4.1 is complete. □

5. Remarks

Finally, we list several remarks.

Remark 5.1. When $y = 1$, the recurrence relations (3.1) and (3.2) becomes

$$F_n = \sum_{r=0}^{n-1} \binom{n}{r} F_r, \quad n \in \mathbb{N}$$

and

$$F_n = (-1)^{n+1} 2 \left[\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} F_k - 1 \right], \quad n \geq 0.$$

These two recurrence relations appeared in [6] in different forms.

Remark 5.2. Let $u = u(t) = y(e^t - 1)$. Then, by the Faà di Bruno formula (4.3) and the identities (4.4) and (4.5), we have

$$\begin{aligned}\lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}\, t^n} \frac{1}{1-y(e^t-1)} &= \lim_{t\to 0} \sum_{k=0}^{n} \frac{\mathrm{d}^k}{\mathrm{d}\, u^k} \frac{1}{1-u} \, \mathrm{B}_{n,k}\left(ye^t, ye^t, \dots, ye^t\right)\\ = \lim_{t\to 0} \sum_{k=0}^{n} \frac{k!}{(1-u)^{k+1}} \left(ye^t\right)^k \mathrm{B}_{n,k}(1,1,\dots,1) &= \sum_{k=0}^{n} S(n,k) k! y^k = F_n(y).\end{aligned}$$

This recovers once again that the Fubini numbers and polynomials F_n and $F_n(y)$ have generating functions $\frac{1}{2-e^t}$ and $\frac{1}{1-y(e^t-1)}$.

References

[1] N. Bourbaki, *Elements of Mathematics: Functions of a Real Variable: Elementary Theory*, Translated from the 1976 French original by Philip Spain. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004; Available online at `http://dx.doi.org/10.1007/978-3-642-59315-4`.

[2] K. N. Boyadzhiev, *A series transformation formula and related polynomials*, Int. J. Math. Math. Sci. **2005**, no. 23, 3849–3866; Available online at `https://doi.org/10.1155/IJMMS.2005.3849`.

[3] N. D. Cahill, J. R. D'Errico, D. A. Narayan, and J. Y. Narayan, *Fibonacci determinants*, College Math. J. **3** (2002), 221–225; Available online at `https://doi.org/10.2307/1559033`.

[4] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974; Available online at `https://doi.org/10.1007/978-94-010-2196-8`.

[5] M. E. Dasef and S. M. Kautz, *Some sums of some importance*, College Math. J. **28** (1997), 52–55.

[6] A. Dil and V. Kurt, *Investigating geometric and exponential polynomials with Euler-Seidel matrices*, J. Integer Seq. **14** (2011), no. 4, Article 11.4.6, 12 pp.
[7] O. A. Gross, *Preferential arrangements*, Amer. Math. Monthly **69** (1962), 4–8; Available online at https://doi.org/10.2307/2312725.
[8] R. D. James, *The factors of a square-free integer*, Canad. Math. Bull. **11** (1968), 733–735; Available online at https://doi.org/10.4153/CMB-1968-089-7.
[9] L. Kargın, *Some formulae for products of Fubini polynomials with applications*, arXiv preprint (2016), available online at https://arxiv.org/abs/1701.01023.
[10] D. S. Kim, T. Kim, H.-I. Kwon, and J.-W. Park, *Two variable higher-order Fubini polynomials*, J. Korean Math. Soc. **55** (2018), no. 4, 975–986; Available online at https://doi.org/10.4134/JKMS.j170573.
[11] F. Qi, *Derivatives of tangent function and tangent numbers*, Appl. Math. Comput. **268** (2015), 844–858; Available online at http://dx.doi.org/10.1016/j.amc.2015.06.123.
[12] F. Qi, *Diagonal recurrence relations, inequalities, and monotonicity related to the Stirling numbers of the second kind*, Math. Inequal. Appl. **19** (2016), no. 1, 313–323; Available online at https://doi.org/10.7153/mia-19-23.
[13] F. Qi and R. J. Chapman, *Two closed forms for the Bernoulli polynomials*, J. Number Theory **159** (2016), 89–100; Available online at http://dx.doi.org/10.1016/j.jnt.2015.07.021.
[14] S. M. Tanny, *On some numbers related to the Bell numbers*, Canad. Math. Bull. **17** (1974/75), no. 5, 733–738; Available online at https://doi.org/10.4153/CMB-1974-132-8.

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